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Forbidden triples for hamiltonicity

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Abstract

In this paper we characterize all triples of connected graphs C, X, Y (where C is a claw) and such that every 2-connected CXY-free graph G is hamiltonian.This result together with a previous result by Faudree, Gould, Jacobson, and Lesniak give a full characterization of triples of forbidden subgraphs implying hamiltonicity of 2-connected graphs. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout the paper, by a *graph* we always mean a finite undirected graph $G = (V(G), E(G))$ without loops and multiple edges. We follow the most common graph-theoretical terminology and notation and for concepts not defined here we refer to [2].

If $M \subset V(G)$ then $\langle M \rangle$ denotes the induced subgraph on M. If $\emptyset \overset{\subset}{\neq} M \overset{\subset}{\neq} V(G)$ we say that $\langle M \rangle$ is a *proper induced subgraph* of G. If $H_1, \ldots, H_k$ are graphs, then we say that a graph G is $H_1 \ldots H_k$-free if G does not contain a copy of any of the graphs $H_1, \ldots, H_k$ as an induced subgraph. The graphs $H_1, \ldots, H_k$ will be referred to in this context as *forbidden subgraphs*.

For some graphs that will often occur as forbidden subgraphs we will use the fixed terminology and notation, see Fig. 1.

We define $\mathscr{P}$ to be the class of graphs obtained by taking two vertex-disjoint triangles $\langle\{v_1, v_2, v_3\}\rangle$, $\langle\{w_1, w_2, w_3\}\rangle$ and by joining every pair of vertices $\{v_i, w_i\}$ by a path P_n for $n \geqslant 3$ or by a triangle. We denote graphs from the class $\mathscr{P}$ by P_{x_1,x_2,x_3}, where $x_i = n$ if v_i, w_i are joined by P_n, and $x_i = T$ if v_i, w_i are joined by a triangle. Graphs from $\mathscr{P}$

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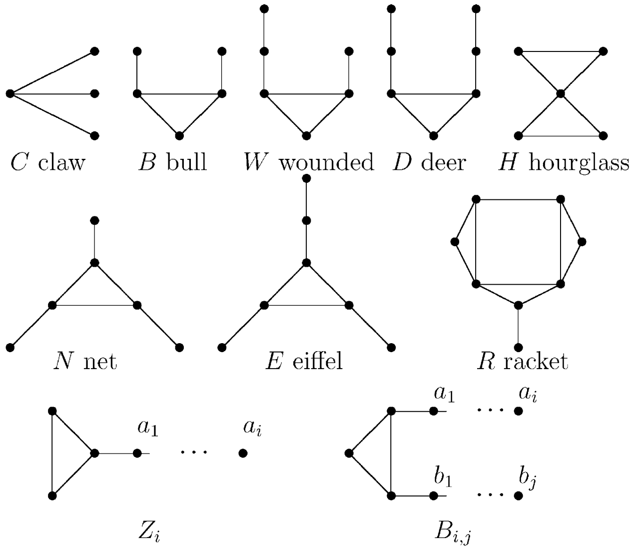

Fig. 1.

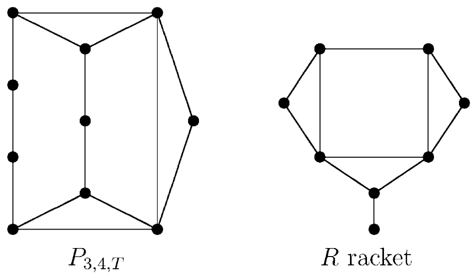

Fig. 2.

play an important role throughout the paper. The *racket* R is the graph obtained by removing one vertex of degree four from a copy of $P_{T,T,T}$ (see Fig. 2).

The following result was proved in [3].

Theorem A (Brousek [3]). *Every* 2*-connected non-hamiltonian claw-free graph contains a graph from the class* $\mathscr{P}$ *as an induced subgraph.*

In [3] are also given as a corollary of Theorem A the following examples of graphs X, Y such that every 2-connected CXY-free graph is hamiltonian.

Theorem B (Brousek [3]). *Let* G *be a* 2*-connected graph satisfying one of the following conditions.*

(i) *G is $CDP_{3,3,3}$-free,*
(ii) *G is $CEP_{T,T,T}$-free,*
(iii) *G is $CP_7P_{T,T,T}$-free.*

Then G is hamiltonian.

Bedrossian [1] characterized all pairs of forbidden graphs which imply hamiltonicity.

Theorem C (Bedrossian [1]). *Let X and Y be connected graphs, both different from P_3, and let G be a 2-connected graph that is not a cycle. Then G being XY-free implies G is hamiltonian if and only if (up to a symmetry) X is a claw and $Y \in \{P_4, P_5, P_6, C_3, Z_1, Z_2, B, N, W\}$.*

Faudree et al. in [4] characterized all triples of graphs X, Y, Z such that none of X, Y, Z is isomorphic to the claw and G being 2-connected and XYZ-free implies hamiltonicity.

Theorem D (Faudree et al. [4]). *Let G be a 2-connected graph of sufficient large order and let X, Y, Z be connected graphs different from P_3 and the claw. Then G being XYZ-free implies F is hamiltonian if and only if X, Y, Z are induced subgraphs of one of the following triples:*

(i) $P_4, K_4 \backslash e, K_{[n/2]+1}$,
(ii) $P_5, K_4 \backslash e, K_{2,3}$,
(iii) $P_6, K_3, K_{2,2}$,
(iv) $C_{1,2,2}, K_3, K_{2,2}$,
(v) $P_4, K_{1,1,3}, K_{2,3}$,
(vi) $P_5, K_3, K_{2,[n/3]}$,
(vii) $C_{1,1,3}, K_3, K_{2,2}$,
(viii) $C_{1,1,2}, K_3, K_{2,[(n-1)/2]-2}$,

where $C_{i,j,k}$ denotes a graph consist of three paths of length $i+1, j+1, k+1$ with one common endvertex.

In the main result of this paper, Theorem 1, we characterize all triples of forbidden subgraphs X, Y, Z which imply hamiltonicity and one of them is the claw. This result together with Theorem D gives a full characterization of forbidden triples of subgraphs for hamiltonicity.

2. Main result

For convenience we will say that a triple of graphs X, Y, Z is a *good triple* if every 2-connected XYZ-free graph is necessarily hamiltonian. We will say a good triple

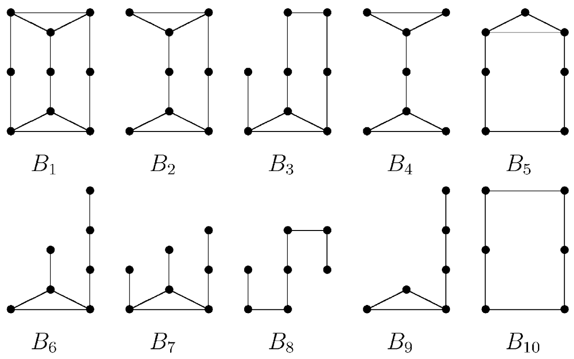

Fig. 3.

X_1, X_2, X_3 is *maximal* if there is no good triple Y_1, Y_2, Y_3 such that $X_i \subset Y_i$ for $i = 1, 2, 3$ (in the sense of induced subgraphs) and $X_i \neq Y_i$ for at least one $i \in \{1, 2, 3\}$ (or, equivalently, if there is no good triple Y_1, Y_2, Y_3 such that the class of $X_1X_2X_3$-free graphs is a proper subclass of the class of $Y_1Y_2Y_3$-free graphs). Remark, that with respect to Theorem C we will consider that no forbidden subgraph is a induced subgraph of P_6, N or W.

Theorem 1. *Let X, Y be connected graphs such that neither X nor Y is an induced subgraph of any of the graphs P_6, N or W. Then*

(i) *There exist* 62 *pairs X, Y of graphs such that the property* "*G is* 2*-connected and CXY-free*" *implies G is hamiltonian.*
(ii) *C, X, Y is a maximal triple of forbidden subgraphs such that G being CXY-free implies G is hamiltonian if and only if $CXY \in \{CDP_{3,3,3}, CEP_{T,T,T}, CP_7P_{T,T,T}, CB_{1,3}R\}$.*

Proof. Let C, X, Y be a good triple. The graphs $P_{3,3,3}$ and $P_{T,T,T}$ are 2-connected, non-hamiltonian and claw-free and thus the set of CXY-free graphs contains neither $P_{3,3,3}$ nor $P_{T,T,T}$. Consequently, the pair X, Y must contain both an induced subgraph of $P_{3,3,3}$ and an induced subgraph of $P_{T,T,T}$. All connected induced subgraphs of the graph $P_{3,3,3}$ which are not induced subgraphs of P_6, N or W are listed in Fig. 3 and, analogously, all connected induced subgraphs of the graph $P_{T,T,T}$ which are not induced subgraphs of P_6, N or W are listed in Fig. 4.

From these lists we can see that $P_{3,3,3}$ and $P_{T,T,T}$ contain no common induced subgraph which is not an induced subgraph of P_6, N or W. This implies that if CXY is a good triple, then (up to symmetry) X must be a graph from Fig. 3 and Y a graph from Fig. 4. Thus at the moment we have 160 candidates for good triples.

Consider a graph $P_{i,j,k}$ such that $i, j, k \geqslant 4$. This graph is 2-connected, non-hamiltonian and claw-free. Moreover it contains no induced subgraph isomorphic to C_4, C_6, H, to two triangles joined by an edge, or B_4. Thus, at least one of the graphs X and Y does

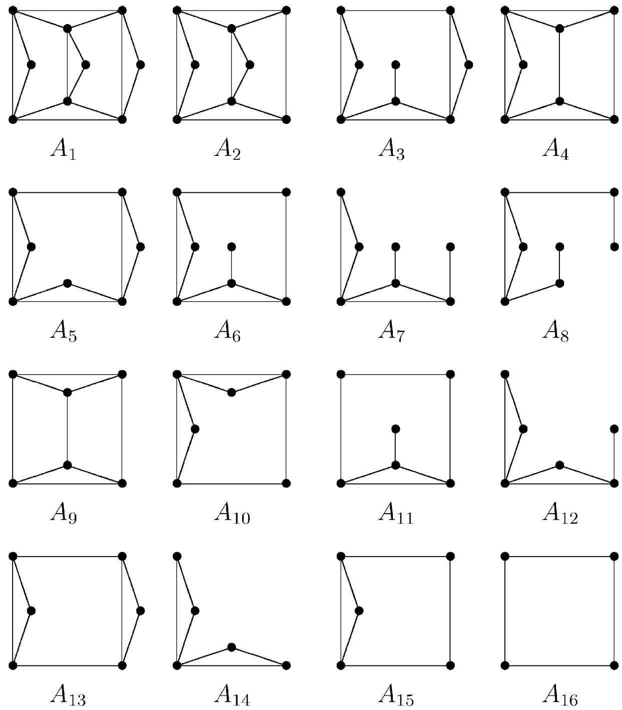

Fig. 4.

not contain a copy of these graphs as an induced subgraph. Among the graphs in Fig. 4 only A_8 (the deer) and among the graphs in Fig. 3 only the graphs B_6, B_7, B_8 and B_9 satisfy this condition.

Let $X = A_8$. Then the triple CA_8B_1 is in fact the triple $CDP_{3,3,3}$ and, by Theorem B, this is a good triple. It is easy to see that this triple is also maximal. From this it immediately follows that all triples CA_8B_i where $i \neq 1$ are good but not maximal. Thus, it remains to consider all pairs $X = A_i, Y = B_j$ such that $i \neq 8$ and $j = 6, 7, 8, 9$.

Now the triple CA_1B_7 is in fact the triple $CEP_{T,T,T}$, the triple CA_1B_8 is the triple $CP_7P_{T,T,T}$ and both these triples are good by Theorem B. It is easy to see that both these triples are also maximal. Thus, all the triples CA_iB_7 and CA_iB_8 for $i \neq 1, 8$ are good but not maximal.

It remains to consider the triples CA_iB_6 and CA_iB_9 for $i \neq 8$. In [5] it was proved that the only 2-connected non-hamiltonian CZ_3-free graphs are the graphs $P_{T,T,T}$ and $P_{T,T,3}$ and it is easy to see that these two graphs are also the only two graphs from the class $\mathscr{P}$ which are B_6-free. If we remove a vertex of degree four from $P_{T,T,T}$ and a vertex of degree three from $P_{T,T,3}$ then in both these cases we get the graph A_3 (which is isomorphic to the racket R). This implies that the triple $CB_{1,3}R$ is a good triple. Hence, all triples CA_iB_j where $j = 6, 9$ and A_i is an induced subgraph of $A_3 = R$ are good but not maximal. These are all graphs from Fig. 4 except A_1, A_2, A_4 and A_9.

Now we have last eight triples CA_iB_j for $i = 1, 2, 4, 9$ and $j = 6, 9$. But the graph $P_{T,T,3}$ contains no such graph as an induced subgraph. Thus, these triples are not good triples and consequently the triple CA_3B_6 is a maximal good triple. The proof is complete. □

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